

BORCHERDS PRODUCTS ON UNITARY GROUP $U(2, 1)$

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ABSTRACT. In this note, we construct canonical bases for the spaces of weakly holomorphic modular forms with poles supported at the cusp ∞ for $\Gamma_0(4)$ of integral weight k for $k \leq -1$, and we make use of the basis elements for the case $k = -1$ to construct explicit Borcherds products on unitary group $U(2, 1)$.

1. INTRODUCTION

In 1998, Borcherds developed a new method to produce memomorphic modular forms on an orthogonal Shimura variety from weakly holomorphic classical modular forms via regularized theta liftings. These memomorphic modular forms have two distinct properties. The first one is the so-called Borcherds product expansion at a cusp of the Shimura variety—his original motivation to prove the Moonshine conjecture. The second is that the divisor of these modular forms are known to be a linear combination of special divisors dictated by the principal part of the input weakly holomorphic forms. The second feature has been extended to produce so-called automorphic green functions for special divisors using harmonic Maass forms via regularized theta lifting by Bruinier ([5] and Bruinier-Funke ([8]), which turned out to be very useful to generalization of the well-known Gross-Zagier formula ([13]) and the beautiful Gross-Zagier factorization formula of singular moduli ([12]) to Shimura varieties of orthogonal type $(n, 2)$ and unitary type $(n, 1)$ (see for example [10], [9], [6], [1], [2], [21], [22]). On the other hand, the Borcherds product expansion and in particular its integral structure is essential to prove modularity of some generating functions of arithmetic divisors on these Shimura varieties ([7], [15]). Borcherds products are also closely related to Mock theta functions (see for example [18] and references there).

We should mention that the analogue of the Borcherds product to unitary Shimura varieties of type $(n, 1)$ has been worked out by Hofmann ([16]). The Borcherds product expansion in the unitary case is a little more complicated as it is a Fourier-Jacobi expansion rather than Fourier expansion. The purpose of this note is to give some explicit examples of these Borcherds product expansion in concrete term. For this reason, we focus on the Picard modular surface associated to the Hermitian lattice $L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$ with Hermitian form

$$\langle x, y \rangle = x_1 \bar{y}_3 + x_3 \bar{y}_1 + x_2 \bar{y}_2.$$

Our inputs are weakly holomorphic modular forms for $\Gamma_0(4)$ of weight -1 , character $\chi_{-4} := (-4)$ which have poles only at the cusp ∞ , which we denote by $M_{-k}^{1, \infty}(\Gamma_0(4), \chi_{-4}^k)$ with $k = 1$. Our first result (Theorem 2.1) is to give a canonical basis $F_{k, m}$ ($m \geq 1$) for the infinitely dimensional vector space for every $k \geq 1$. The even k case was given by Haddock and Jenkins in [14] in a slightly different fashion. Since $\Gamma_0(4)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$, a simple conjugation also gives a canonical basis for the space of weakly holomorphic forms of $\Gamma_0(4)$ with weight $-k$, character χ_{-4}^k , and having poles only at cusp 0 (resp. $\frac{1}{2}$).

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Next, we use standard induction procedure to produce vector valued weakly modular forms for $\mathrm{SL}_2(\mathbb{Z})$ using our lattice L which will be used to construct Picard modular forms on $U(2, 1)$ (described above). Although the resulting vector valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$ from the three different scalar valued spaces $M_{-k}^{!,P}(\Gamma_0(4), \chi_{-4}^k)$, $P = \infty, 0, \frac{1}{2}$ are linearly independent, they don't generate the whole space. We find it interesting. This concludes Part I of our note, which should be of independent interest.

In Part II, we focus on the unitary group $U(2, 1)$ associated to the above Hermitian form and give explicit Borchers product expansion of the Picard modular forms constructed from $F_m = F_{1,m}$. The delicate part is to choose a proper Weyl chamber, which is a dimensional 3 real manifold and described it explicitly and carefully. Our main formula is Theorem 3.4. We remark that the same method also applies to high dimensional unitary Shimura varieties of unitary type $(n, 1)$ using forms in $M_{1-n}^{!,P}(\Gamma_0(4), \chi_{-4}^k)$ where P is a cusp for $\Gamma_0(4)$. We restrict to $U(2, 1)$ for being as explicit as possible.

2. PART I: VECTOR VALUED MODULAR FORMS

In this part, we derive canonical basis for the space $M_{-k}^{!,\infty}(\Gamma_0(4), \chi_{-4}^k)$ for any integer $k \geq 0$, and investigate the properties of the vector valued modular forms arising from $M_{-k}^{!,\infty}(\Gamma_0(4), \chi_{-4}^k)$. For completeness, we will also give canonical basis for $M_{-k}^{!,0}(\Gamma_0(4), \chi_{-4}^k)$ and $M_{-k}^{!,\frac{1}{2}}(\Gamma_0(4), \chi_{-4}^k)$.

2.1. Canonical Basis for $M_{-k}^{!,\infty}(\Gamma_0(4), \chi_{-4}^k)$.

Let $\chi_{-4}(\cdot) := \left(\frac{-4}{\cdot}\right)$ be the Kronecker symbol modulo 4. Recall that $X_0(4)$ has 3 cusps, represented by ∞ , 0, and $\frac{1}{2}$. For each cusp P , let $M_{-k}^{!,P}(\Gamma_0(4), \chi_{-4}^k)$ denote the space of weakly holomorphic modular forms, which are holomorphic everywhere except at the cusp P , of weight $-k$ on $\Gamma_0(4)$ with character χ_{-4}^k . We will focus mainly on the cusp ∞ and will remark on other cusps (very similar) in the end. We will also denote $M_{-k}^!(\Gamma_0(4), \chi_{-4}^k)$ for the space of weakly holomorphic modular forms for $\Gamma_0(4)$ of weight $-k$ and character χ_{-4}^k .

Let τ be a complex number with positive imaginary part, and set $q = e(\tau) = e^{2\pi i \tau}$, and $q_r = e^{2\pi i \tau / r}$. The Dedekind eta function is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Throughout this paper, we write η_m for $\eta(m\tau)$. The well known Jacobi theta functions are defined by

$$\vartheta_{00}(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \vartheta_{01}(\tau) = \sum_{n=-\infty}^{\infty} (-q)^{n^2}, \quad \vartheta_{10}(\tau) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}.$$

Now we define three functions as follows.

$$(2.1) \quad \theta_1 = \theta_1(\tau) := \frac{1}{16} \vartheta_{10}^4(\tau) = \frac{\eta_4^8}{\eta_2^4} = q + O(q^2),$$

$$(2.2) \quad \theta_2 = \theta_2(\tau) := \vartheta_{00}^2(\tau) = \frac{\eta_2^{10}}{\eta_1^4 \eta_4^4} = 1 + O(q),$$

$$(2.3) \quad \varphi_{\infty} = \varphi_{\infty}(\tau) := \left(\frac{\eta_1}{\eta_4}\right)^8 = q^{-1} + O(1).$$

Here are some basic facts [14] about the functions θ_1 , θ_2 and φ_{∞} .

- (1) $\theta_1(\tau)$ is a holomorphic modular form of weight 2 on $\Gamma_0(4)$ with trivial character, has a simple zero at the cusp ∞ , and vanishes nowhere else.
- (2) $\theta_2(\tau)$ is a holomorphic modular form of weight 1 on $\Gamma_0(4)$ with character χ_{-4} , has a zero of order $\frac{1}{2}$ at the irregular cusp $\frac{1}{2}$, and vanishes nowhere else.
- (3) $\varphi_\infty(\tau)$ is a modular form of weight 0 on $\Gamma_0(4)$ with trivial character, has exactly one simple pole at the cusp ∞ and a simple zero at the cusp 0.

Theorem 2.1. (1) For $k \geq 1$ odd, there is a (canonical) basis $F_{k,m}$ ($m \geq 1$) of $M_{-k}^{!,\infty}(\Gamma_0(4), \chi_{-4})$ whose Fourier expansion has the following property:

$$F_{k,m} = q^{-\frac{k+1}{2}-m+1} + \sum_{n \geq -\frac{k-1}{2}} c(n)q^n.$$

(2) For $k > 1$ even, there is a (canonical) basis $F_{k,m}$ ($m \geq 1$) of $M_{-k}^{!,\infty}(\Gamma_0(4))$ whose Fourier expansion has the following property:

$$F_{k,m} = q^{-\frac{k}{2}-m+1} + \sum_{n \geq -\frac{k}{2}+1} c(n)q^n,$$

Proof. We prove (1) first. Notice that $X_0(4)$ has no elliptic points [11, Section 3.9]. For $F \in M_{-k}^{!,\infty}(\Gamma_0(4), \chi_{-4})$, the valence formula for $\Gamma_0(4)$ asserts that

$$\sum_{z \in \Gamma_0(4) \backslash \mathbb{H}} \text{ord}_z(F) + \text{ord}_\infty(F) + \text{ord}_0(F) + \text{ord}_{1/2}(F) = -\frac{k}{2}.$$

This implies $\text{ord}_{1/2} F \geq \frac{1}{2}$ ($1/2$ is the unique irregular cusp), $\text{ord}_\infty(F) \leq -\frac{k+1}{2}$. This implies the uniqueness of the basis $\{F_{k,m}\}$ if it exists. We prove the existence by inductively construct a sequence of monic polynomials $P_{k,m}(x)$ of degree m ($m \geq 0$) such that $F_{k,m+1} = \theta_2 \theta_1^{-\frac{k+1}{2}} P_{k,m}(\varphi_\infty)$ give the basis we seek, i.e., with the following property

$$(2.4) \quad F_{k,m+1} = \theta_2 \theta_1^{-\frac{k+1}{2}} P_{k,m}(\varphi_\infty) = q^{-\frac{k+1}{2}-m} + \sum_{n \geq -\frac{k-1}{2}} c(n)q^n.$$

(The awkward notation $F_{k,m+1}$ instead of $F_{k,m}$ will be clear in last section.)

(1) Notice that $\theta_2 \theta_1^{-\frac{k+1}{2}} \in M_{-k}^{!,\infty}(\Gamma_0(4), \chi_{-4})$ with

$$\theta_2 \theta_1^{-\frac{k+1}{2}} = q^{-\frac{k+1}{2}} + \sum_{n \geq -\frac{k-1}{2}} c(n)q^n.$$

So we can and will first define $P_{k,0} = 1$.

(2) For $m \geq 1$, assume that $P_{k,m-1}(x) \in \mathbb{C}[x]$ is constructed with degree $m-1$, leading coefficient 1, and the property

$$F_{k,m} = \theta_2 \theta_1^{-\frac{k+1}{2}} P_{k,m-1}(\varphi_\infty) = q^{-\frac{k+1}{2}-m+1} + \sum_{n \geq -\frac{k-1}{2}} c(n)q^n.$$

Then it is easy to see

$$F_{k,m} \varphi_\infty = q^{-\frac{k+1}{2}-m} + \sum_{n > -\frac{k+1}{2}-m} d(n)q^n.$$

Let

$$P_{k,m} = xP_{k,m-1} - \sum_{n=-\frac{k+1}{2}-m+1}^{-\frac{k+1}{2}} d(n)P_{k,-n},$$

and

$$F_{k,m+1} = \theta_2 \theta_1^{-\frac{k+1}{2}} P_{k,m}(\varphi_\infty).$$

Then $F_{k,m+1}$ satisfies (2.4). By induction, we prove the existence of the basis $\{F_{k,m}\}$, and (1).

The proof of (2) is similar and is left to the reader. The basis $\{F_{k,m+1}\}$, $m \geq 0$, has the form

$$(2.5) \quad F_{k,m+1} = \theta_1^{-\frac{k}{2}} Q_{k,m}(\varphi_\infty) = q^{-\frac{k}{2}-m} + \sum_{n=-\frac{k}{2}+1}^{\infty} c(n)q^n$$

for a unique monic polynomial $Q_{k,m}$ of degree m . □

Remark 2.2. The canonical basis given in Theorem 2.1(2) was given in a slightly different form first by Haddock and Jenkins [14].

The following corollary follows directly from the proof of Theorem 2.1(1).

Corollary 2.3. Every weakly holomorphic modular form $f(\tau) \in M_{-k}^{!,\infty}(\Gamma_0(4), \chi_{-4}^k)$ with k odd, vanishes at cusp $1/2$.

2.2. Vector Valued Modular Form Arising from $M_{-k}^{!,\infty}(\Gamma_0(4), \chi_{-4}^k)$.

Let L be an even lattice over \mathbb{Z} with symmetric non-degenerate bilinear form (\cdot, \cdot) and associated quadratic form $Q(x) = \frac{1}{2}(x, x)$. Let L' be the dual lattice of L . Assume that L has rank $2m+2$ and signature $(2m, 2)$. Then the Weil representation of the metaplectic group $\text{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$ factors through $\text{SL}_2(\mathbb{Z})$. Thus we have a unitary representation ρ_L of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$, defined by

$$\begin{aligned} \rho_L(T)\phi_\mu &= e(-Q(\mu))\phi_\mu, \\ \rho_L(S)\phi_\mu &= \frac{\sqrt{i}^{2m-2}}{\sqrt{|L'/L|}} \sum_{\beta \in L'/L} e((\mu, \beta))\phi_\beta \end{aligned}$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, ϕ_γ for $\mu \in L'/L$ are the standard basis elements of $\mathbb{C}[L'/L]$ and $e(z) = e^{2\pi iz}$. We remark that the Weil representation ρ_L depends only on the finite quadratic module $(L'/L, Q)$ (called the discriminant group of L), where $Q(x+L) = Q(x) \pmod{1} \in \mathbb{Q}/\mathbb{Z}$.

Let k be an integer and \vec{F} be a $\mathbb{C}[L'/L]$ valued function on \mathbb{H} and let $\rho = \rho_L$ be a representation of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$. For $\gamma \in \text{SL}_2(\mathbb{Z})$ we define the slash operator by

$$\left(\vec{F} \Big|_{k,\rho} \gamma \right) (\tau) = (c\tau + d)^{-k} \rho(\gamma)^{-1} \vec{F}(\gamma\tau),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on \mathbb{H} via $\gamma\tau = \frac{a\tau+b}{c\tau+d}$.

Definition 2.4. Let k be an integer. A function $\vec{F} : \mathbb{H} \rightarrow \mathbb{C}[L'/L]$ is called a weakly holomorphic vector valued modular form of weight k with respect to $\rho = \rho_L$ if it satisfies

(1) $\vec{F}|_{k,\rho} \gamma = F$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

(2) \vec{F} is holomorphic on \mathbb{H} ,

(3) \vec{F} is meromorphic at the cusp ∞ .

The space of such forms is denoted by $M_{k,\rho}^!$.

The invariance of T -action implies that $\vec{F} \in M_{k,\rho}^!$ has a Fourier expansion of the form

$$\vec{F} = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c(n, \phi_\mu) q^n \phi_\mu.$$

Note that $c(n, \phi_\mu) = 0$ unless $n \equiv -Q(\mu) \pmod{1}$.

From now on, we focus on the special case with the discriminant group $L'/L \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with quadratic form $Q(x, y) = \frac{1}{4}(x^2 + y^2) \pmod{1}$. For our purpose (in last section), it is convenient to identify $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}[i]/2\mathbb{Z}[i]$, where $Q(z) = \frac{1}{4}z\bar{z} \in \mathbb{Q}/\mathbb{Z}$. We write ϕ_0, ϕ_1, ϕ_i and ϕ_{1+i} for the basis elements of $\mathbb{C}[L'/L]$ corresponding to $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ respectively.

Let $F = F(\tau) \in M_{-k}^{!,\infty}(\Gamma_0(4), \chi_{-4})$ with k odd and positive. Then using $\Gamma_0(4)$ -lifting, we can construct a vector valued modular form $\vec{F} = \vec{F}(\tau)$ arising from $F(\tau)$ as follows:

$$(2.6) \quad \vec{F}(\tau) = \sum_{\gamma \in \Gamma_0(4) \backslash \mathrm{SL}_2(\mathbb{Z})} (F|_{-k} \gamma) \rho_L(\gamma)^{-1} \phi_0 = \frac{1}{2} \sum_{\gamma \in \Gamma_1(4) \backslash \mathrm{SL}_2(\mathbb{Z})} (F|_{-k} \gamma) \rho_L(\gamma)^{-1} \phi_0.$$

Define modular forms F_0, F_2 and F_3 as follows. Let

$$F|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} a(n) q_4^n.$$

Then for $j \in \{0, 2, 3\}$, we write

$$F_j = \sum_{n=0}^{\infty} a(4n+j) q_4^{4n+j}.$$

We also define $F_{1/2}$ to be

$$F_{1/2} = F|_{-k} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \sum_{n=0}^{\infty} b(n) q_2^n.$$

Then a simple calculation gives

$$(2.7) \quad \vec{F}(\tau) = (-2iF_0 + F) \phi_0 - 2iF_3 \phi_1 - 2iF_3 \phi_i + (-2iF_2 - F_{1/2}) \phi_{1+i}.$$

The following theorem gives some basic facts about F_0, F_2, F_3 and $F_{1/2}$.

Theorem 2.5. *With the above definitions, we have*

$$(2.8) \quad F_0 \in M_{-k}^!(\Gamma_0(4), \chi_{-4}),$$

$$(2.9) \quad F_3 \in M_{-k}^!(\Gamma_0(4), \chi_1)$$

where $\chi_1(\gamma) = \chi_{-4}(d)e(-ab/4)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

$$(2.10) \quad (2iF_2 + F_{1/2}) \in M_{-k}^!(\Gamma_0(4), \chi_2)$$

where $\chi_2(\gamma) = \chi_{-4}(d)e(-ab/2)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

and

$$(2.11) \quad F_{1/2} \in M_{-k}^!(\delta^{-1}\Gamma_0(4)\delta, \chi_{-4})$$

$$\text{where } \delta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Proof. By (2.7), and [19, Section 3, p. 6] or [20, Proposition 4.5], we can show that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

$$(2.12) \quad (-2iF_0 + F)|_{-k} \gamma = \chi_{-4}(d)(-2iF_0 + F),$$

$$(2.13) \quad F_3|_{-k} \gamma = \chi_{-4}(d)e(-ab/4)F_3,$$

$$(2.14) \quad (-2iF_2 - F_{1/2})|_{-k} \gamma = \chi_{-4}(d)e(-ab/2)(-2iF_2 - F_{1/2}).$$

Since $F \in M_{-k}^!(\Gamma_0(4), \chi_{-4})$, then (2.12) implies (2.8). Relations (2.9) and (2.10) follow directly from (2.13) and (2.14), respectively. The last relation (2.11) follows from the definition of $F_{1/2}$,

$$F_{1/2} = F|_{-k} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

□

Theorem 2.6. *Let k be odd. Let $F = F(\tau) \in M_{-k}^{!,\infty}(\Gamma_0(4), \chi_{-4})$ with*

$$F(\tau) = \sum_{n=-m}^{\infty} c(n)q^n.$$

Write

$$F|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} a(n)q_4^n \quad \text{and} \quad F|_{-k} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \sum_{n=0}^{\infty} b(n)q_2^n.$$

And let the $\Gamma_0(4)$ -lifting of F be

$$\vec{F}(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c(n, \phi_\mu) q^n \phi_\mu.$$

Then we have

(i)

$$\begin{aligned} c(n, \phi_0) &= -2ia(4n) + c(n), \\ c(n, \phi_1) &= c(n, \phi_i) = -2ia(4n), \\ c(n, \phi_{1+i}) &= -2ia(4n) - b(2n), \end{aligned}$$

(ii) the principal part of the vector valued modular form $\vec{F}(\tau)$ is

$$(c(-m)q^{-m} + \cdots + c(-1)q^{-1}) \phi_0,$$

(iii) the constant term of the ϕ_0 -component of $\vec{F}(\tau)$ is

$$c(0, \phi_0) = -(8i)^{k+1} \sum_{n=\frac{k+1}{2}}^m c(-n) P_{k, n-\frac{k+1}{2}}(0) + c(0).$$

In particular, when $k = 1$, the constant term of the ϕ_0 -component of $\vec{F}(\tau)$ is

$$(2.15) \quad c(0, \phi_0) = \sum_{n=1}^m c(-n) \left(\sum_{d|n} (64\chi_{-4}(n/d) + 4\chi_{-4}(d)) d^2 \right).$$

Proof. Assertion (i) follows directly from (2.7). For the assertion (ii), since F is holomorphic at 0 and $\frac{1}{2}$, then F_j for $j \in \{0, 2, 3\}$ and $F_{1/2}$ will not contribute anything to the principal part of \vec{F} , and thus by (2.7) the principal part of \vec{F} is

$$(c(-m)q^{-m} + \cdots + c(-1)q^{-1}) \phi_0.$$

For the assertion (iii), we first note by (i) that

$$c(0, \phi_0) = -2ia(0) + c(0).$$

By Theorem 2.1(1), we have

$$(2.16) \quad F = c(-m)\theta_2\theta_1^{-\frac{k+1}{2}}P_{k,m-\frac{k+1}{2}}(\varphi_\infty) + \cdots + c\left(-\frac{k+1}{2}\right)\theta_2\theta_1^{-\frac{k+1}{2}}P_{k,0}(\varphi_\infty)$$

We can verify that

$$\theta_2\theta_1^{-\frac{k+1}{2}}\varphi_\infty^l \Big|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = O(q^{\frac{l}{4}}),$$

and thus $\theta_2\theta_1^{-\frac{k+1}{2}}\varphi_\infty^l \Big|_{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ will not contribute anything to the constant term of F_0 when $l \geq 1$. Therefore,

$$\begin{aligned} a(0) &= \left(\sum_{n=\frac{k+1}{2}}^m c(-n)P_{k,n-\frac{k+1}{2}}(0)\theta_2\theta_1^{-\frac{k+1}{2}} \Big|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)_0 \\ &= -(8i)^{k+1} \sum_{n=\frac{k+1}{2}}^m c(-n)P_{k,n-\frac{k+1}{2}}(0) \end{aligned}$$

where $(f)_0$ denote the constant term of the q -expansion of f . Hence, we have

$$c(0, \phi_0) = -(8i)^{k+1} \sum_{n=\frac{k+1}{2}}^m c(-n)P_{k,n-\frac{k+1}{2}}(0) + c(0).$$

For (2.15), according to (iii), we need to show that

$$P_{1,m}(0) = \sum_{d|(m+1)} \chi_{-4}((m+1)/d)d^2 \quad \text{and} \quad c(0) = \sum_{n=1}^m c(-n) \left(4 \sum_{d|n} \chi_{-4}(d)d^2 \right).$$

For the first formula, we first observe that

$$\theta_2\theta_1^{-1}\varphi_\infty^\ell = q^{-\ell-1} + \sum_{j=1}^{\ell} c_\ell(-j)q^{-j} + O(1)$$

for $0 \leq \ell \leq m$. Thus there are b_1, \dots, b_{m-1} such that

$$h(\tau) := \theta_2\theta_1^{-1}\varphi_\infty^m + b_{m-1}\theta_2\theta_1^{-1}\varphi_\infty^{m-1} + \cdots + b_1\theta_2\theta_1^{-1}\varphi_\infty = q^{-m-1} + a(-1)q^{-1} + O(1)$$

for some constant $a(-1)$. Let $g(\tau)$ be defined by

$$g(\tau) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_{-4}(n/d) d^2 \right) q^n = \sum_{n=1}^{\infty} d_n q^n.$$

It is known [17] that $g(\tau)$ is a weight 3 modular form on $\Gamma_0(4)$ with character χ_{-4} . We note by the basic facts about θ_1 , θ_2 and φ_{∞} that $h(\tau)$ vanishes at the cusps $1/2$ and 0 . Then by [4, Theorem 3.1], we have

$$d_{m+1} + a(-1) = 0, \text{ i.e., } d_{m+1} = -a(-1).$$

Therefore

$$P_{1,m}(0) = d_{m+1} = \sum_{d|(m+1)} \chi_{-4}((m+1)/d) d^2.$$

This proves the first formula. For the second one, the proof is similar by noting that

$$h_1(\tau) := \theta_2 \theta_1^{-1} P_{1,m}(\varphi_{\infty}) = q^{-m-1} + C + O(q)$$

and

$$g_1(\tau) = 1 + 4 \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi_{-4}(d) d^2 \right) q^n$$

is [17] a weight 3 modular form on $\Gamma_0(4)$ with character χ_{-4} . Then again [4, Theorem 3.1] shows that

$$C = 4 \sum_{d|(m+1)} \chi_{-4}(d) d^2.$$

This together with (2.16) proves the second formula. □

Example 2.7. Let $k = 1$ and $F(\tau) = \theta_2 \theta_1^{-1} = \frac{\eta_2^{14}}{\eta_1^4 \eta_4^{12}} \in M_{-1}^{1,\infty}(\Gamma_0(4), \chi_{-4})$. Then we have

$$(2.17) \quad \vec{F}(\tau) = (-2iF_0 + F) \phi_0 - 2iF_3 \phi_1 - 2iF_3 \phi_i + (-2iF_2 - F_{1/2}) \phi_{1+i}$$

where F_0 , F_2 , F_3 and $F_{1/2}$ are defined as follows; suppose

$$\begin{aligned} F|_{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= 32i \frac{\eta(\tau/2)^{14}}{\eta(\tau/4)^4 \eta(\tau)^{12}} \\ &= 32i (1 + 12q^{1/4} + 76q^{2/4} + 352q^{3/4} + 1356q + 4600q^{5/4} \\ &\quad + 14176q^{6/4} + 40512q^{7/4} + \dots) \\ &= 32i (1 + 1356q + O(q^2)) \\ &\quad + 32i (12q^{1/4} + 4600q^{5/4} + O(q^{9/4})) \\ &\quad + 32i (76q^{2/4} + 14176q^{6/4} + O(q^{10/4})) \\ &\quad + 32i (352q^{3/4} + 40512q^{7/4} + O(q^{11/4})), \end{aligned}$$

then

$$\begin{aligned} F_0 &= 32i (1 + 1356q + O(q^2)), \\ F_2 &= 32i (76q^{2/4} + 14176q^{6/4} + O(q^{10/4})) \\ F_3 &= 32i (352q^{3/4} + 40512q^{7/4} + O(q^{11/4})). \end{aligned}$$

And

$$F_{1/2} = F|_{-1} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 64 \left(q^{1/2} - 8q^{3/2} + 42q^{5/2} + O(q^{7/2}) \right).$$

From (2.17), we note that the principal part of F is $e(-\tau)\phi_0$ and the constant term of the ϕ_0 -component is $c(0, \phi_0) = 68$.

2.3. Canonical Basis for $M_{-k}^{1,0}(\Gamma_0(4), \chi_{-4}^k)$ and $M_{-k}^{1,\frac{1}{2}}(\Gamma_0(4), \chi_{-4}^k)$.

We complete this section by giving canonical basis for the other two companions of $M_{-k}^{1,\infty}(\Gamma_0(4), \chi_{-4}^k)$. Let $\theta_3(\tau)$, $\varphi_0(\tau)$ and $\varphi_{1/2}(\tau)$ be defined by

$$(2.18) \quad \theta_3 = \theta_3(\tau) := \vartheta_{01}^4(\tau) = \frac{\eta_1^8}{\eta_2^4} = 1 + O(q),$$

$$(2.19) \quad \varphi_0 = \varphi_0(\tau) := \left(\frac{\eta_4}{\eta_1} \right)^8 = q + O(q^2),$$

$$(2.20) \quad \varphi_{1/2} = \varphi_{1/2}(\tau) := \frac{\eta_1^8 \eta_4^{16}}{\eta_2^{24}} = q + O(q^2).$$

Here are some basic facts about θ_3 , φ_0 and $\varphi_{1/2}$:

- (1) $\theta_3(\tau)$ is a weight 2 modular form on $\Gamma_0(4)$ with trivial character, has a simple zero at the cusp 0, and vanishes nowhere else;
- (2) $\varphi_0(\tau)$ is a weight 0 modular form on $\Gamma_0(4)$ with trivial character, has a simple pole at the cusp 0 and a simple zero at the cusp ∞ , and vanishes nowhere else;
- (3) $\varphi_{1/2}(\tau)$ is a weight 0 modular form on $\Gamma_0(4)$ with trivial character, has a simple pole at the cusp $\frac{1}{2}$ and a simple zero at the cusp ∞ , and vanishes nowhere else.

Theorem 2.8. *Let θ_2 , θ_3 and φ_0 be as defined in (2.2), (2.18) and (2.19), respectively.*

- (1) *For k odd, the set $\{\theta_2 \theta_3^{-\frac{k+1}{2}} P_{k,m}(\varphi_0)\}_{m=0}^{\infty}$ where $P_{k,m}$ is a monic polynomial of degree m such that*

$$\theta_2 \theta_3^{-\frac{k+1}{2}} P_{k,m}(\varphi_0) \Big|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = q_4^{-\frac{k+1}{2}-m} + \sum_{n=-\frac{k-1}{2}}^{\infty} c(n) q_4^n,$$

is a canonical basis for $M_{-k}^{1,0}(\Gamma_0(4), \chi_{-4})$.

- (2) *For k even, the set $\{\theta_3^{-\frac{k}{2}} P_{k,m}(\varphi_0)\}_{m=0}^{\infty}$ where $P_{k,m}$ is a monic polynomial of degree m such that*

$$\theta_3^{-\frac{k}{2}} P_{k,m}(\varphi_0) \Big|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = q_4^{-\frac{k}{2}-m} + \sum_{n=-\frac{k}{2}+1}^{\infty} c(n) q_4^n,$$

is a canonical basis for $M_{-k}^{1,0}(\Gamma_0(4))$.

Theorem 2.9. *Let θ_2 and $\varphi_{1/2}$ be as defined in (2.2) and (2.20), respectively. Then the set $\{\theta_2^{-k} P_{k,m}(\varphi_{1/2})\}_{m=0}^{\infty}$ where $P_{k,m}$ is a monic polynomial of degree m such that*

$$\theta_2^{-k} P_{k,m}(\varphi_{1/2}) \Big|_{-k} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = q^{-\frac{k}{2}-m} + \sum_{n=-\frac{k}{2}+1}^{\infty} c(n) q^n,$$

is a canonical basis for $M_{-k}^{1,\frac{1}{2}}(\Gamma_0(4), \chi_{-4}^k)$.

Proofs of Theorems 2.8 and 2.9 are similar to that of Theorem 2.1, so we omit the details.

Remark 2.10. For a cusp P , denote by $M_{-k,\rho_L}^{!,P}$ the space of vector valued modular forms induced from $M_{-k}^{!,P}(\Gamma_0(4), \chi_{-4}^k)$ via $\Gamma_0(4)$ -lifting. We have, by (2.7),

$$M_{-k,\rho_L}^{!,\infty} + M_{-k,\rho_L}^{!,0} + M_{-k,\rho_L}^{!,\frac{1}{2}} = M_{-k,\rho_L}^{!,\infty} \oplus M_{-k,\rho_L}^{!,0} \oplus M_{-k,\rho_L}^{!,\frac{1}{2}}.$$

Clearly, $M_{-k,\rho_L}^{!,\infty} + M_{-k,\rho_L}^{!,0} + M_{-k,\rho_L}^{!,\frac{1}{2}}$ is a subspace of $M_{-k,\rho_L}^!$. In general, the former space may not be equal to the latter one. We first note that by (2.7) every vector valued modular form in $M_{-k,\rho_L}^{!,\infty} + M_{-k,\rho_L}^{!,0} + M_{-k,\rho_L}^{!,\frac{1}{2}}$ must have the same component functions at ϕ_1 and ϕ_i . We now give an example of functions in $M_{-1,\rho_L}^!$ that does not have this property. Let $F(\tau) = \theta_2 \theta_1^{-1} \in M_{-1}^{!,\infty}(\Gamma_0(4), \chi_{-4})$. Then as above we write the $\Gamma_0(4)$ -lifting of $F(\tau)$ as

$$\vec{F}(\tau) = (-2iF_0 + F)\phi_0 - 2iF_3\phi_1 - 2iF_3\phi_i + (-2iF_2 - F_{1/2})\phi_{1+i}$$

where

$$F_j = \sum_{n=0}^{\infty} a(4n+j)q_4^{4n+j},$$

$$F|_{-k} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} a(n)q_4^n$$

and

$$F_{1/2} = F|_{-1} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

By (2.9), we know that $F_3(\tau) \in M_{-1}^!(\Gamma_1(4), \chi)$ where $\chi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = e(-b/4)$. Now we do $\Gamma_1(4)$ -lifting on $F_3(\tau)$ against ϕ_1 and get

$$\vec{F}_3(\tau) = -4if_0\phi_0 + (2F_3 + 4if_3)\phi_1 + (-4if_3 - 2f_{1/2})\phi_i + 4if_2\phi_{1+i}$$

where

$$f_j = \sum_{\substack{n \in \mathbb{Z} \\ n \gg -\infty}} \tilde{a}(4n+j)q_4^{4n+j},$$

$$F_3|_{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sum_{\substack{n \in \mathbb{Z} \\ n \gg -\infty}} \tilde{a}(n)q_4^n$$

and

$$f_{1/2} = F_3|_{-1} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Now the component functions at ϕ_1 and ϕ_i are $2F_3 + 4if_3$ and $-4if_3 - 2f_{1/2}$, respectively. We can compute and verify that they are not the same. Therefore, $\vec{F}_3(\tau)$ is not in the space $M_{-k,\rho_L}^{!,\infty} + M_{-k,\rho_L}^{!,0} + M_{-k,\rho_L}^{!,\frac{1}{2}}$.

3. PART II: BORCHERDS PRODUCTS ON $U(2, 1)$

It is well-known that the vector valued weakly modular forms construction in Part I can be used to construct memomorphic modular forms on Shimura varieties of orthogonal type $(n, 2)$ and unitary type $(n, 1)$ with Borchers product formula and known divisors. In this part, we focus on one special case to make it very explicitly—the Picard modular surfaces over $\mathbf{k} = \mathbb{Q}(i)$. In particular, we describe a Weyl chamber explicit and write down the Borchers product expression concretely.

This part is devoted to deriving Borchers products lifted from a vector valued modular form arising from $M_{-1}^{!,\infty}(\Gamma_0(4), \chi_{-4})$.

3.1. Picard modular surfaces over $\mathbf{k} = \mathbb{Q}(i)$. Let $(V, \langle \cdot, \cdot \rangle)$ be a Hermitian vector space over \mathbf{k} of signature $(2, 1)$ and let $H = U(V)$. Let $V_{\mathbb{C}} = V \otimes_{\mathbf{k}} \mathbb{C}$, and

$$\mathcal{L} = \{w \in V_{\mathbb{C}} : \langle w, w \rangle < 0\}.$$

Then $\mathcal{K} = \mathcal{L}/\mathbb{C}^\times$ is the Hermitian domain for $H(\mathbb{R})$, and \mathcal{L} is the tautological line bundle over \mathcal{K} . For a congruence subgroup Γ of $H(\mathbb{Q})$, the associated Picard modular surface $X_\Gamma = \Gamma \backslash \mathcal{K}$ is defined over some number field.

Given an isotropic line $\mathbf{k}\ell$ (i.e., a cusp), choose another isotropic element ℓ' with $(\ell, \ell') \neq 0$. Let $V_0 = (\mathbf{k}\ell + \mathbf{k}\ell')^\perp$, and let

$$\mathcal{H} = \mathcal{H}_{\ell, \ell'} = \{(\tau, \sigma) \in \mathbb{H} \times V_{0, \mathbb{C}} \mid \operatorname{Im} \tau > \frac{\langle \sigma, \sigma \rangle}{4|\langle \ell', \ell \rangle|^2}\}.$$

Then the map

$$(3.1) \quad \mathcal{H} \rightarrow \mathcal{L}, \quad (\tau, \sigma) \mapsto z(\tau, \sigma) = 2i\langle \ell', \ell \rangle \tau \ell + \sigma + \ell'$$

gives rise to an isomorphism $\mathcal{H} \cong \mathcal{K}$. It is also a nowhere vanishing section of the line bundle \mathcal{L} . Using this map, we can define action of $H(\mathbb{R})$ on \mathcal{H} and automorphy factor $j(\gamma, \tau, \sigma)$ via the equation

$$(3.2) \quad \gamma z(\tau, \sigma) = j(\gamma, \tau, \sigma) z(\gamma(\tau, \sigma)).$$

Definition 3.1. Let Γ be a unitary modular group. A holomorphic automorphic form of weight k and with character χ for Γ is a function $g : \mathcal{H} \rightarrow \mathbb{C}$, with the following properties:

- (1) g is holomorphic on \mathcal{H} ,
- (2) $g(\gamma(\tau, \sigma)) = j(\gamma; \tau, \sigma)^k \chi(\gamma) g(\tau, \sigma)$ for all $\gamma \in \Gamma$.

We remark that a holomorphic modular form g for Γ is automatically holomorphic at cusps.

Now we make everything concrete and explicit. First choose a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of V with Gram matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so $V = \oplus \mathbf{k}\mathbf{e}_i \cong \mathbf{k}^3$ with Hermitian form

$$(3.3) \quad \langle x, y \rangle = x_1 \bar{y}_3 + x_2 \bar{y}_2 + x_3 \bar{y}_1 = {}^t x J \bar{y},$$

and

$$H = H(\mathbb{Q}) = \{h \in \operatorname{GL}_3(\mathbf{k}) \mid h J {}^t \bar{h} = J\}.$$

We take the lattice

$$L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$$

(instead of the typical $\mathbb{Z}[i]^3$). Its \mathbb{Z} -dual lattice is

$$L' = \{v \in V \mid \text{Tr}_{\mathbf{k}/\mathbb{Q}} \langle v, L \rangle \subset \mathbb{Z}\} = \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$$

So $L'/L \cong \frac{1}{2}\mathbb{Z}[i]/\mathbb{Z}[i]$ with quadratic form $Q(x) = x\bar{x} \in \frac{1}{4}\mathbb{Z}/\mathbb{Z}$, which is the same finite quadratic module considered in Part I. Let

$$\begin{aligned} U(L) &= \{g \in H \mid gL = L\} \\ &= H \cap \left\{ \begin{pmatrix} \mathbb{Z}[i] & \mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \mathbb{Z}[i] & \mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \frac{1}{2}\mathbb{Z}[i] & \frac{1}{2}\mathbb{Z}[i] & \mathbb{Z}[i] \end{pmatrix} \right\}. \end{aligned}$$

be the stabilizer of L in H , and Γ_L be the subgroup of $U(L)$ which acts on the discriminant group L'/L trivially:

$$\Gamma_L = U(L) \cap \left\{ \begin{pmatrix} \mathbb{Z}[i] & 2\mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \mathbb{Z}[i] & 1 + 2\mathbb{Z}[i] & 2\mathbb{Z}[i] \\ \mathbb{Z}[i] & 2\mathbb{Z}[i] & \mathbb{Z}[i] \end{pmatrix} \right\}.$$

Take the cusp $\ell = \mathbf{e}_1$ and $\ell' = \mathbf{e}_3$. Then $V_0 \cong \mathbf{k}$ with Hermitian form $\langle x, y \rangle = x\bar{y}$, and

$$\mathcal{H} = \{(\tau, \sigma) \in \mathbb{H} \times \mathbb{C} \mid 4\text{Im}(\tau) > |\sigma|^2\}.$$

Moreover, one has for $\gamma = (a_{ij}) \in H$

$$\gamma(\tau, \sigma) = \left(\frac{a_{11}\tau + (2i)^{-1}a_{12}\sigma + (2i)^{-1}a_{13}}{2ia_{31}\tau + a_{32}\sigma + a_{33}}, \frac{2ia_{21}\tau + a_{22}\sigma + a_{23}}{2ia_{31}\tau + a_{32}\sigma + a_{33}} \right).$$

and

$$j(\gamma, \tau, \sigma) = \frac{\langle \gamma z, \ell \rangle}{\langle \ell', \ell \rangle} = 2i\tau a_{31} + a_{32}\sigma + a_{33}.$$

Let P_ℓ be the stabilizer of the cusp $\mathbf{k}\ell$ in H . Then $P_\ell = N_\ell M_\ell$ with

$$\begin{aligned} M_\ell &= \{m(a, b) = \text{Diag}(a, b, \bar{a}^{-1}) \mid a \in \mathbf{k}^\times, b \in \mathbf{k}^1\}, \\ N_\ell &= \{n(b, c) = \begin{pmatrix} 1 & -2\bar{b} & -2b\bar{b} + 2ic \\ 0 & 1 & 2b \\ 0 & 0 & 1 \end{pmatrix} \mid b \in \mathbf{k}, c \in \mathbb{Q}\}, \end{aligned}$$

where $\mathbf{k}^1 = \{a \in \mathbf{k} \mid a\bar{a} = 1\}$ is the norm one group. Notice that N_ℓ is a Heisenberg group actin on $\mathcal{H}_{\ell, \ell'}$ via

$$n(b, c)(\tau, \sigma) = (\tau + c + i\bar{b}(\sigma + b), \sigma + b).$$

In particular

$$n(0, c)(\tau, \sigma) = (\tau + c, \sigma).$$

Let

$$\Gamma_{L, \ell} = \Gamma_L \cap N_\ell = \{n(b, c) : b \in \mathbb{Z}[i], c \in \mathbb{Z}\}.$$

Then for a holomorphic modular form $f(\tau, \sigma)$ for Γ_L , we have partial Fourier expansion at the cusp $\mathbf{k}\ell$:

$$(3.4) \quad f(\tau, \sigma) = \sum_{n \geq 0} f_n(\sigma) q^n.$$

3.2. The Hermitian Space V as a Quadratic Space. As mentioned in the previous subsection, the hermitian space V can be viewed as a quadratic space $V_{\mathbb{Q}}$ of signature $(4,2)$ associated with bilinear form induced from the hermitian form:

$$(x, y) = \text{Tr}_{\mathbf{k}/\mathbb{Q}} \langle x, y \rangle.$$

Then the lattice L can be considered as a quadratic \mathbb{Z} -lattice in $V_{\mathbb{Q}}$. Denote by

$$\text{SO}(V_{\mathbb{Q}}) = \{g \in \text{SL}(V_{\mathbb{Q}}) | (gx, gy) = (x, y) \text{ for all } x, y \in V_{\mathbb{Q}}\}$$

the special orthogonal group of $V_{\mathbb{Q}}$ and its set of real points as $\text{SO}(V_{\mathbb{Q}})(\mathbb{R}) \cong \text{SO}(4, 2)$. A model for the symmetric domain of $\text{SO}(V_{\mathbb{Q}})(\mathbb{R})$ is the Grassmannian of two-dimensional negative definite subspaces of $V_{\mathbb{Q}}$, denoted by Gr_O . It can be realized as a tube domain \mathcal{H}_O as follows. Denote by $V_{\mathbb{Q}}(\mathbb{C})$ the complex quadratic space $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ with (\cdot, \cdot) extended to a \mathbb{C} -valued bilinear form.

Now we view L as a \mathbb{Z} -lattice. Let $e_1 \in L$ be a primitive isotropic lattice vector and choose an isotropic dual vector $e_2 \in L'$ with $(e_1, e_2) = 1$. Denote by K the Lorentzian \mathbb{Z} -sublattice $K = L \cap e_1^{\perp} \cap e_2^{\perp}$ with respect to (\cdot, \cdot) . The tube domain model \mathcal{H}_O is one of the two connected components of the following subset of $K \otimes_{\mathbb{Z}} \mathbb{C}$

$$\{Z = X + iY | X, Y \in K \otimes_{\mathbb{Z}} \mathbb{R}, Q(Y) < 0\}.$$

Recall that $\ell = \mathbf{e}_1$ and $\ell' = \mathbf{e}_3$. We define

$$e_1 = \ell, e_2 = \frac{\hat{1}}{2}\ell', e_3 = -\hat{i}\ell, e_4 = -\frac{\hat{i}}{2}\ell'$$

where we denote by $\hat{\mu}$ the endomorphism of $V_{\mathbb{Q}}(\mathbb{R})$ induced from the scalar multiplication with μ . Then we can check that $\{e_1, e_2, e_3, e_4\}$ is a basis for $(\mathbb{Z}[i]\ell + \mathbb{Z}[i]\ell') \otimes_{\mathbb{Z}} \mathbb{Q}$ and we can see that $K \otimes_{\mathbb{Z}} \mathbb{R} = ((\mathbb{Q}e_3 + \mathbb{Q}e_4) \otimes_{\mathbb{Z}} \mathbb{R}) \oplus (V_0 \otimes_{\mathbb{Z}} \mathbb{R})$. Thus we can identify Y with $y_1e_3 + y_2e_4 + \sigma \in K \otimes_{\mathbb{Z}} \mathbb{R}$. Now denote by \mathcal{C} the set of $Y = y_1e_3 + y_2e_4 + \sigma$ with $y_1y_2 + Q(\sigma) < 0$, $y_1 < 0$ and $y_2 > 0$. We can fix \mathcal{H}_O as the component for which $Y \in \mathcal{C}$. Therefore, $\mathcal{H}_O = K \otimes_{\mathbb{Z}} \mathbb{R} + i\mathcal{C}$.

In addition, the tube domain \mathcal{H}_O can be mapped biholomorphically to any one of the two connected components of a negative cone of $\mathbb{P}^1(V_{\mathbb{Q}})(\mathbb{C})$ given by

$$\{[Z_L] | (Z_L, Z_L) = 0, (Z_L, \bar{Z}_L) < 0\}.$$

We fix this component and denote it by \mathcal{K}_O . For each $[Z_L]$, we can uniquely represent it as

$$Z_L = e_2 - q(Z)e_1 + Z$$

with $Z \in \mathcal{H}_O$.

3.3. Embedding of \mathcal{H} into \mathcal{H}_O . As in [16, Section 4], we can embed \mathcal{H} into \mathcal{H}_O via

$$(3.5) \quad (\tau, \sigma) \rightarrow \iota(\tau, \sigma) = -\tau e_3 + ie_4 + \mathfrak{z}(\sigma)$$

where

$$(3.6) \quad \mathfrak{z}(\sigma) = \frac{\hat{1}}{2}\sigma + i \left(-\frac{\hat{i}}{2} \right) \sigma.$$

Similarly, \mathcal{K}_U can be embedded into \mathcal{K}_O through the identifications between \mathcal{K}_U and \mathcal{H} , and between \mathcal{K}_O and \mathcal{H}_O . Namely,

$$(3.7) \quad z = \ell' + 2i\tau\ell + \sigma \rightarrow Z_L = -i\tau e_1 + e_2 - \tau e_3 + ie_4 + \mathfrak{z}(\sigma).$$

3.4. Weyl Chambers of $K \otimes_{\mathbb{Z}} \mathbb{R}$. In Theorem 2.1 (1), we have shown that $F_{1,m} = q^{-m} + O(1)$ for $m \geq 1$, form a canonical basis for $M_{-1}^{1,\infty}(\Gamma_0(4), \chi_{-4})$. Therefore, to study the Borcherds product lifted from $M_{-1,\rho_L}^{1,\infty}$, it suffices to start with $F_{1,m}$. Since we only deal with weight -1 in the rest of this paper, we will simply write $F_m = F_{1,m}$, and $\vec{F}_m = \vec{F}_{1,m}$.

For general definitions of the following, we refer the reader to [5, Chapter 3.1]. For $\kappa \in K$ with $q(\kappa) > 0$, denote by κ^\perp the orthogonal complement of κ in $K \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by \mathcal{D}_K the Grassmannian of negative 1-lines of $K \otimes_{\mathbb{Z}} \mathbb{R}$, which can be realized as

$$\begin{aligned}\mathcal{D}_K &= \{\mathbb{R}w \subset K_{\mathbb{R}} \mid q(w) < 0\} \\ &\cong \{w = y_1 e_3 + e_4 + (y_3 + i y_4) \mid y_i \in \mathbb{R}, q(w) < 0\}\end{aligned}$$

Then by considering the Grassmannian of negative 1-lines of κ^\perp , it corresponds to a codimension 1 sub-manifold of the Grassmannian \mathcal{D}_K of $K \otimes_{\mathbb{Z}} \mathbb{R}$.

In our case, a Heegner divisor of index $(m, 0)$, $H_K(m, 0)$, is a locally finite union of codimension 1 sub-manifolds of \mathcal{D}_K , namely,

$$H_K(m, 0) = \{z \in \mathcal{D}_K \mid \exists \kappa \in K \text{ with } q(\kappa) = m \text{ and } (z, \kappa) = 0\}$$

Let $\vec{F}_m(\tau)$ be the vector valued modular form arising from F_m . It is known by Theorem 2.6 that the principal part of $\vec{F}_m(\tau)$ is $q^{-m}\phi_0$. The Weyl chambers attached to $\vec{F}_m(\tau)$ are the connected components W_m of

$$\mathcal{D}_K - H_K(m, 0).$$

Fix a Weyl chamber W_m of \mathcal{D}_K , we can also define the corresponding Weyl chambers of $K \otimes_{\mathbb{Z}} \mathbb{R}$ and \mathcal{H} by

$$\begin{aligned}W_{m,K} &= \{w \in K \otimes_{\mathbb{Z}} \mathbb{R} \mid \mathbb{R}w \in W_m\}, \\ W_{m,U} &= \left\{ (\tau, \sigma) \in \mathcal{H} \mid \text{Im}(\iota(\tau, \sigma)) = -\text{Im} \tau e_3 + e_4 - \frac{\hat{i}}{2} \sigma \in W_{m,K} \right\},\end{aligned}$$

respectively. In the following lemma, we give explicit description of the Weyl chamber that we use to construct Borcherds product in Theorem 3.4.

Lemma 3.2. (1) *Let*

$$\begin{aligned}(3.8) \quad W_m &= \left\{ y_1 e_3 + e_4 + (y_3 + i y_4) \in \mathcal{D}_K \mid \begin{array}{l} y_1 < r^2 + s^2 - m + 2r y_3 + 2s y_4 \quad \forall r, s \in \mathbb{Z}, \\ 1 + 2t y_3 + 2h y_4 > 0 \quad \forall t, h \in \mathbb{Z}, t^2 + h^2 = m, \\ t y_3 + h y_4 > 0, \quad \forall t, h \in \mathbb{Z}, t^2 + h^2 = m, t > 0, \\ y_4 > 0 \text{ if } m \text{ is a square.} \end{array} \right\} \\ &\subset \left\{ y_1 e_3 + e_4 + (y_3 + i y_4) \in \mathcal{D}_K \mid \begin{array}{l} k_2 y_1 < -k_1 + 2k_3 y_3 + 2k_4 y_4 \quad \forall k_i \in \mathbb{Z}, k_2 > 0, k_1 k_2 + k_3^2 + k_4^2 = m, \\ k + 2t y_3 + 2h y_4 > 0 \quad \forall k, t, h \in \mathbb{Z}, k > 0, t^2 + h^2 = m, \\ t y_3 + h y_4 > 0, \quad \forall t, h \in \mathbb{Z}, t^2 + h^2 = m, t > 0, \\ y_4 > 0 \text{ if } m \text{ is a square} \end{array} \right\}.\end{aligned}$$

Then W_m is a Weyl chamber containing e_3 .

(2) *Let*

$$K_m = \left\{ \lambda = \lambda_1 e_3 - \lambda_2 e_4 + \frac{1}{2}(\lambda_3 + i \lambda_4) \in K' \mid \begin{array}{l} Q(\lambda) = m \text{ and } \lambda_3, \lambda_4 \in 2\mathbb{Z}, \\ \text{or } Q(\lambda) \leq 0, \\ (\lambda, W_m) > 0 \end{array} \right\}$$

where $(\lambda, W_m) > 0$ means that $(\lambda, w) > 0$ for all $w \in W_m$. Then

$$K_m = \left\{ \lambda = \lambda_1 e_3 - \lambda_2 e_4 + \frac{1}{2}(\lambda_3 + i\lambda_4) \left| \begin{array}{l} \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{Z} \\ \lambda_2 > 0, \\ \text{or } \lambda_2 = 0 \text{ and } \lambda_1 > 0, \\ \text{or } \lambda_2 = \lambda_1 = 0 \text{ and } \lambda_3 > 0, \\ \text{or } \lambda_2 = \lambda_1 = \lambda_3 = 0 \text{ and } \lambda_4 > 0 \end{array} \right. \right\}.$$

Proof. For Assertion (1), it is clear that W_m contains e_3 since the set of (y_3, y_4, y_1) determined by the inequalities in W_m contains $y_1 = -\infty$. We only need to show W_m is actually a Weyl chamber.

Write $\kappa = \kappa_1 e_3 + \kappa_2 e_4 + \kappa_3 + i\kappa_4 \in K$ with $\kappa_i \in \mathbb{Z}$. Since $(-\kappa)^\perp = \kappa^\perp$, we can assume $k_2 \geq 0$. By the definition of Weyl chamber W_m , we can see that a Weyl chamber W_m can be viewed as a connected component of \mathbb{R}^3 cut out by the planes

$$k_2 y_1 + k_1 + 2k_3 y_3 + 2k_4 y_4 = 0$$

for all $k_1, \dots, k_4 \in \mathbb{Z}$ with $k_2 \geq 0$ and $k_1 k_2 + k_3^2 + k_4^2 = m$.

When $k_2 = 0$ and m is representable by sum of two squares, then we have planes

$$k_1 + 2k_3 y_3 + 2k_4 y_4 = 0$$

perpendicularly passing through the $y_3 - y_4$ plane. In this case, the connected components are determined by the connected components of the $y_3 - y_4$ plane cut out by the lines

$$k_1 + 2k_3 y_3 + 2k_4 y_4 = 0,$$

and it is easy to find that one of the connected components \mathcal{C}_1 can be identified as

$$\left\{ (y_3, y_4) \in \mathbb{R}^2 \left| \begin{array}{l} 1+2ty_3+2hy_4 > 0 \ \forall t, h \in \mathbb{Z}, t^2+h^2=m, \\ ty_3+hy_4 > 0, \ \forall t, h \in \mathbb{Z}, t^2+h^2=m, t>0, \\ y_4 > 0 \text{ if } m \text{ is a square} \end{array} \right. \right\}$$

which is a subset of

$$\left\{ (y_3, y_4) \in \mathbb{R}^2 \left| \begin{array}{l} k+2ty_3+2hy_4 > 0 \ \forall k, t, h \in \mathbb{Z}, k>0, t^2+h^2=m, \\ ty_3+hy_4 > 0, \ \forall t, h \in \mathbb{Z}, t^2+h^2=m, t>0, \\ y_4 > 0 \text{ if } m \text{ is a square} \end{array} \right. \right\}.$$

When $k_2 > 0$, with the aid of MAPLE, we can check that there is a connected component \mathcal{C}_2 of \mathbb{R}^3 covered by

$$y_1 = r^2 + s^2 - m + 2ry_3 + 2sy_4$$

for $r, s \in \mathbb{Z}$. Such a connected component contains $y_1 < -m$, and all the other planes

$$k_2 y_1 = -k_1 + 2k_3 y_3 + 2k_4 y_4$$

for $k_1, \dots, k_4 \in \mathbb{Z}$ with $k_2 > 0$ and $k_1 k_2 + k_3^2 + k_4^2 = m$. In conclusion, $W_m = \mathcal{C}_1 \cap \mathcal{C}_2$ is a connected component of \mathbb{R}^3 cut out by the planes

$$k_2 y_1 + k_1 + 2k_3 y_3 + 2k_4 y_4 = 0$$

for all $k_1, \dots, k_4 \in \mathbb{Z}$ with $k_2 \geq 0$ and $k_1 k_2 + k_3^2 + k_4^2 = m$, and thus W_m is a Weyl chamber.

Now let us prove Assertion (2).

- (i) Suppose that $Q(\lambda) = m$ and $\lambda_3, \lambda_4 \in 2\mathbb{Z}$. By (3.8), we note that $y_1 e_3 + e_4 + (y_3 + iy_4) \in W_m$ implies that

$$k_2 y_1 < -k_1 + 2k_3 y_3 + 2k_4 y_4$$

for all $k_i \in \mathbb{Z}$ with $k_2 > 0$ and $k_1 k_2 + k_3^2 + k_4^2 = m$, which is equivalent to that

$$k_2 y_1 + k_1 + 2k_3 y_3 + 2k_4 y_4 > 0$$

for all $k_i \in \mathbb{Z}$ with $k_2 < 0$ and $k_1 k_2 + k_3^2 + k_4^2 = m$. Therefore, when $\lambda_2 \neq 0$ and $Q(\lambda) = m$, that is, $\lambda_1(-\lambda_2) + \frac{1}{4}(\lambda_3^2 + \lambda_4^2) = m$, $(\lambda, W_m) > 0$ if and only if $-\lambda_2 < 0$, that is, $\lambda_2 > 0$. Similarly, by the other conditions given in (3.8), we can conclude that when $Q(\lambda) = m$, $(\lambda, W_m) > 0$ if

and only if $\lambda_2 < 0$, or $\lambda_2 = 0$ and $\lambda_1 > 0$, or $\lambda_2 = \lambda_1 = 0$ and $\lambda_3 > 0$, or $\lambda_2 = \lambda_1 = \lambda_3 = 0$ and $\lambda_4 > 0$.

(ii) Now suppose that $Q(\lambda) \leq 0$, that is, $\lambda_1\lambda_2 + \frac{1}{4}(\lambda_3^2 + \lambda_4^2) \leq 0$. By (3.8), we know that

$$y_1 < r^2 + s^2 - m + 2ry_3 + 2sy_4$$

for all $r, s \in \mathbb{Z}$. By [5, Lemma 3.2], it is known that if $(\lambda, w_0) > 0$ for a $w_0 \in W_m$, then $(\lambda, W_m) > 0$. Thus $(\lambda, W_m) > 0$ if and only if $\lambda_2 > 0$. When $\lambda_2 = 0$, since $Q(\lambda) \leq 0$, then $\lambda_3 = \lambda_4 = 0$, and thus $(\lambda, w) = \lambda_1$ for $w \in W_m$. This implies that $(\lambda, W_m) > 0$ if and only if $\lambda_1 > 0$ when $\lambda_2 = 0$.

□

3.5. The Weyl Vector for \vec{F}_m . For the Weyl chamber W_m described in (3.8), using [3, Theorem 13.3] (see also [22, Theorem 2.2] for minor correction) or [5, Theorem 3.22], we can compute the corresponding Weyl vector of \vec{F}_m :

$$\rho(W, \vec{F}_m) = \rho_{e_3}e_3 + \rho_{e_4}e_4 + \rho$$

with

$$\begin{aligned} \rho_{e_3} &= -\frac{1}{6} \sum_{d|m} (16\chi_{-4}(m/d) + \chi_{-4}(d)) d^2 - \frac{1}{24} \sigma_{\chi_{-4}}(m), \\ \rho_{e_4} &= \frac{1}{6} \left[\sigma_{\chi_{-4}}(m) - 6\sigma_1(m) - 24 \left(\sum_{\substack{k+l=m \\ k,l \geq 1}} \sigma_{\chi_{-4}}(k) \sigma_1(l) \right) \right. \\ &\quad \left. + \sum_{d|m} (16\chi_{-4}(m/d) + \chi_{-4}(d)) d^2 \right], \\ \rho &= -(0, \sum_{n=1}^l t_n, 0) \end{aligned}$$

where $\sigma_{\chi_{-4}}(m) = \sum_{d|m} \chi_{-4}(d)$, $0 \leq t_n$ is the t -component of the integral solution of $t^2 + h^2 = m$, $l = \sigma_{\chi_{-4}}(m)$.

3.6. Heegner Divisors for Γ_L . Let $\lambda \in L'$ be a lattice vector with positive norm, i.e., $\langle \lambda, \lambda \rangle > 0$. The complement of λ in \mathcal{K}_U is a closed analytic subset of codimension 1, which we denote as follows.

$$\mathbf{H}(\lambda) = \{[z] \in \mathcal{K}_U \mid \langle z, \lambda \rangle = 0\}.$$

By identification between \mathcal{K}_U and \mathcal{H} , $\mathbf{H}(\lambda)$ can also be considered as a closed analytic subset of \mathcal{H} , and we call such set a prime Heegner divisor on \mathcal{H} . Given $\beta \in L'/L$ and $m \in \mathbb{Z}_{>0}$, a Heegner divisor of index (m, β) in \mathcal{H} is defined as the locally finite sum

$$\mathbf{H}(m, \beta) = \sum_{\substack{\lambda \in \beta + L \\ Q(\lambda) = m}} \mathbf{H}(\lambda).$$

The associated Heegner divisor in $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ is $\mathbf{Z}(m, \beta) = \Gamma_L \backslash \mathbf{H}(m, \beta)$.

3.7. Borchers Products. In this section, we give a family of new Borchers products explicitly by using the results of Hofmann [16, Thm. 4, Thm. 5 and Cor. 1]. We first summarize Hofmann's results as follows.

Theorem 3.3 (Hofmann). *Let \mathbb{F} be an imaginary quadratic field. Let L be an even hermitian lattice of signature $(m, 1)$ with $m \geq 1$, and $\ell \in L$ a primitive isotropic vector. Let $\ell' \in L'$ an isotropic vector with $\langle \ell, \ell' \rangle \neq 0$. Further assume that L is the direct sum of a hyperbolic plane $H \cong \mathcal{O}_{\mathbb{F}} \oplus \partial_{\mathbb{F}}^{-1}$ and a definite part D with $\langle D, H \rangle = 0$.*

Given a weakly holomorphic form of weight $f \in M_{1-m, \rho_L}^!$ with Fourier coefficients $c(n, \beta)$ satisfying $c(n, \beta) \in \mathbb{Z}$ for $n < 0$, there is a meromorphic function $\Psi(\tau, \sigma; f)$ on \mathcal{H} with the following properties:

- (1) $\Psi(\tau, \sigma; f)$ is an automorphic form of weight $c(0, \phi_0)/2$ for Γ_L with some multiplier system χ of finite order.
- (2) The zeros and poles of $\Psi(\tau, \sigma; f)$ lie on Heegner divisors. The divisor of $\Psi(\tau, \sigma; f)$ on $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ is given by

$$\text{div}(\Psi(\tau, \sigma; f)) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{\substack{n \in \mathbb{Z} - Q(\beta) \\ n > 0}} c(-n, \phi_\beta) \mathbf{H}(n, \beta).$$

The multiplicities of $\mathbf{H}(n, \beta)$ are 2 if $2\beta = 0$ in L'/L , and 1 otherwise.

- (3) For a Weyl chamber W whose closure containing the cusp $\mathbb{Q}e_3$, $\Psi(\tau, \sigma; f)$ has an infinite product expansion of the form

$$\Psi(\tau, \sigma; f) = Ce \left(\frac{\langle z, \rho(W, f) \rangle}{\langle \ell, \ell' \rangle} \right) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \left[1 - e \left(\frac{\langle z, \lambda \rangle}{\langle \ell, \ell' \rangle} \right) \right]^{c(-Q(\lambda), \lambda)},$$

where $z = z(\tau, \sigma) = \ell' + \delta \langle \ell, \ell' \rangle \tau \ell + \sigma$, δ is the square root of the discriminant of \mathbb{F} , the constant C has absolute value 1 and $\rho(W, f)$ is the Weyl vector attached to W and f .

- (4) The lifting is multiplicative: $\Psi(\tau, \sigma; f + g) = \Psi(\tau, \sigma; f) \Psi(\tau, \sigma; g)$.
- (5) Let W be a Weyl chamber such that the cusp corresponding to ℓ is contained in the closure of W . If this cusp is neither a pole nor a zero of $\Psi(\tau, \sigma; f)$, then we have

$$\lim_{\tau \rightarrow \infty} \Psi(\tau, \sigma; f) = Ce (\rho(\bar{W}, f)_\ell) \prod_{\substack{\lambda \in K' \\ \lambda = \frac{1}{2} \kappa \delta \ell \\ \kappa \in \mathbb{Q}_{>0}}} \left(1 - e \left(-\frac{1}{2} \kappa \bar{\delta} \right) \right)^{c(0, \lambda)}$$

where $\rho(W, f)_\ell$ denotes the ℓ -component of the Weyl vector $\rho(W, f)$.

By specializing Theorem 3.3 in our case, we obtain the main result of this note.

Theorem 3.4. *Let $L = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}[i]$ with respect to the standard basis over $\mathbb{Z}[i]$ with hermitian form defined in (3.3). We set $\ell = (1, 0, 0)$ and $\ell' = (0, 0, 1)$. Let \vec{F}_m be the vector valued modular form arising from $F_m = \theta_2 \theta_1^{-1} P_{1, m-1}(\varphi_\infty)$. Then there is a meromorphic function $\Psi(\tau, \sigma; F_m) = \Psi(\tau, \sigma; \vec{F}_m)$ on \mathcal{H} with the following properties:*

- (1) $\Psi(\tau, \sigma; \vec{F}_m)$ is an automorphic form of weight

$$32 \sum_{d|m} \chi_{-4}(n/d) d^2 + 2 \sum_{d|m} \chi_{-4}(d) d^2$$

for Γ_L , with some multiplier system χ of finite order.

(2) The zeros and poles of $\Psi(\tau, \sigma; \vec{F}_m)$ lie on Heegner divisors. The divisor of $\Psi(\tau, \sigma; \vec{F}_m)$ on $X_{\Gamma_L} = \Gamma_L \backslash \mathcal{H}$ is given by

$$\operatorname{div}(\Psi(\tau, \sigma; \vec{F}_m)) = \mathbf{Z}(m, 0) = \Gamma_L \backslash \mathbf{H}(m, 0),$$

where

$$\mathbf{H}(m, 0) = \sum_{\substack{(r_1, s_1, r_2, s_2, r_3, s_3) \in \mathbb{Z}^6 \\ r_1 r_3 + s_1 s_3 + r_2^2 + s_2^2 = m}} \left\{ (\tau, \sigma) \in \mathcal{H} \left| \begin{array}{l} r_1 + 2r_2 \operatorname{Re} \sigma + 2s_2 \operatorname{Im} \sigma + s_3 \operatorname{Re} \tau - r_3 \operatorname{Im} \tau = 0, \\ s_1 + 2r_2 \operatorname{Im} \sigma - 2s_2 \operatorname{Re} \sigma + s_3 \operatorname{Im} \tau + r_3 \operatorname{Re} \tau = 0 \end{array} \right. \right\}.$$

(3) For the Weyl chamber W_m described in (3.8), $\Psi(\tau, \sigma; \vec{F}_m)$ has an infinite product expansion near the cusp $\mathbb{Q}e_3$ (precisely, when $(\tau, \sigma) \in W_{m,U}$ with $\operatorname{Im} \tau$ sufficiently large):

$$(3.9) \quad \Psi(\tau, \sigma; F_m) = A_1(\tau, \sigma) A_2(\sigma) A_3(\sigma) A_4(\sigma) A_5(\tau, \sigma),$$

where

(i)

$$A_1(\tau, \sigma) = e(i\rho_{e_3} - \rho_{e_4}\tau + \bar{\rho}\sigma)$$

where ρ_{e_3} , ρ_{e_4} and ρ are as defined in Subsection 3.5,

(ii)

$$A_2(\sigma) = \begin{cases} [1 - e(-i\sigma\sqrt{m})] & \text{if } m \text{ is a square,} \\ 1 & \text{otherwise,} \end{cases}$$

(iii)

$$A_3(\sigma) = \prod_{\substack{(k_3, k_4) \in \mathbb{Z}_{>0}^2 \\ k_3^2 + k_4^2 = m}} [1 - e(\sigma(k_3 + ik_4))] [1 - e(\sigma(k_3 - ik_4))],$$

(iv)

$$\begin{aligned} A_4(\sigma) &= \prod_{\substack{n_3, n_4 \in \mathbb{Z} \\ n_3^2 + n_4^2 = m}} \prod_{n_2 \in \mathbb{Z}_{>0}} [1 - e(in_2) e(\sigma(n_3 - in_4))] \\ &\times \prod_{n_2 \in \mathbb{Z}_{>0}} (1 - e(in_2))^{c(0,0)} \end{aligned}$$

with

$$c(0, 0) = c(0, \phi_0) = \sum_{d|m} (64\chi_{-4}(m/d) + 4\chi_{-4}(d)) d^2,$$

(v)

$$A_5(\tau, \sigma) = \prod_{\substack{(n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 \\ n_1 > 0}} \left[1 - e\left(n_1\tau + \sigma\left(\frac{n_3}{2} - i\frac{n_4}{2}\right) + in_2\right) \right]^{c(n_1 n_2 - \frac{1}{4}(n_3^2 + n_4^2), \phi_{\vec{n}})}$$

with $\vec{n} = n_2 e_3 - n_1 e_4 + \frac{1}{2}(n_3 + in_4)$.

(4) If the cusp corresponding to ℓ is neither a pole nor a zero of $\Psi(\tau, \sigma; \vec{F}_m)$, then we have

$$\lim_{\tau \rightarrow i\infty} \Psi(\tau, \sigma; \vec{F}_m) = e(i\rho_{e_3}) \prod_{k=1}^{\infty} (1 - e(ki))^{c(0, \phi_0)}$$

where

$$\rho_{e_3} = -\frac{1}{6} \sum_{d|m} (16\chi_{-4}(m/d) + \chi_{-4}(d)) d^2 - \frac{1}{24} \sigma_{\chi_4}(m)$$

as defined in subsection 3.5, and

$$c(0, \phi_0) = \sum_{d|m} (64\chi_{-4}(m/d) + 4\chi_{-4}(d)) d^2$$

as in Theorem 2.6.

Proof. Assertions (1) and (2) follows directly from Theorem 3.3 (1) and (2), respectively.

Then by Theorem 3.3 (3) together with Lemma 3.8, we have that $\Psi(\tau, \sigma; \vec{F}_m)$ has the following infinite product expansion near the cusp $\mathbb{Q}e_3$

$$\begin{aligned} & \psi(\tau, \sigma; \vec{F}_m) \\ &= e(i\rho_{e_3} - \rho_{e_4}\tau + \bar{\rho}\sigma) \\ & \times \prod_{\substack{(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{Z}^4 \\ \lambda_2 > 0, \\ \text{or } \lambda_2 = 0 \text{ and } \lambda_1 > 0, \\ \text{or } \lambda_2 = \lambda_1 = 0 \text{ and } \lambda_3 > 0, \\ \text{or } \lambda_2 = \lambda_1 = \lambda_3 = 0 \text{ and } \lambda_4 > 0.}} \left[1 - e \left(\lambda_2 \tau + \sigma \left(\frac{\lambda_3}{2} - i \frac{\lambda_4}{2} \right) + i \lambda_1 \right) \right]^{c(\lambda_1 \lambda_2 - \frac{1}{4}(\lambda_3^2 + \lambda_4^2), \phi_\lambda)} \end{aligned}$$

where $\lambda = \lambda_1 e_3 - \lambda_2 e_4 + \frac{1}{2}(\lambda_3 + i\lambda_4)$, and ρ_{e_3} , ρ_{e_4} and ρ are as defined in Subsection 3.5. We first set $A_1(\tau, \sigma) = e(i\rho_{e_3} - \rho_{e_4}\tau + \bar{\rho}\sigma)$. Then by decomposing the infinite product according to the four cases in its product index set, we can easily rewrite it as (3.9).

Finally, for Assertion (4), we first note that in our case, $K' = \mathbb{Z}i \oplus \mathbb{Z}[i] \oplus \frac{1}{2}\mathbb{Z}i$ and $\delta = 2i$, then $\lambda \in K'$ and $\lambda = \frac{1}{2}\kappa\delta\ell = \kappa i\ell$ with $\kappa \in \mathbb{Q}_{>0}$ imply that $\kappa \in \mathbb{Z}_{>0}$ and $c(0, \lambda) = c(0, \phi_0)$. Together with the Weyl vector attached to W_m and \vec{F}_m shown in Subsection 3.5, Theorem 3.3 (5) proves Assertion (4). □

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